

# EVALUATING THE EXACT INFINITESIMAL VALUES OF AREA OF SIERPINSKI'S CARPET AND VOLUME OF Menger'S SPONGE

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## Abstract

Very often traditional approaches studying dynamics of self-similarity processes are not able to give their quantitative characteristics at infinity and, as a consequence, use limits to overcome this difficulty. For example, it is well known that the limit area of Sierpinski's carpet and volume of Menger's sponge are equal to zero. It is shown in this paper that recently introduced infinite and infinitesimal numbers allow us to use exact expressions instead of limits and to calculate exact infinitesimal values of areas and volumes at various points at infinity even if the chosen moment of the observation is infinitely faraway on the time axis from the starting point. It is interesting that traditional results that can be obtained without the usage of infinite and infinitesimal numbers can be produced just as finite approximations of the new ones. The importance of the possibility to have this kind of quantitative characteristics for  $E$ -Infinity theory is emphasized.

**Key Words:** Sierpinski's carpet, Menger's sponge, infinite and infinitesimal numbers, area, volume.

## 1 Introduction

Appearance of new powerful approaches modelling the spacetime by fractals (see [1, 2, 4, 7, 8, 9, 10, 23] and references given therein) urges development of adequate mathematical tools allowing one to study fractal objects quantitatively after  $n$  steps executed in a fractal process for both finite and *infinite*  $n$ . Very well developed theories of fractals (see e.g., [4, 5, 6, 12, 13] and references given therein) allow us to give certain numerical answers to questions regarding fractals (calculation of,

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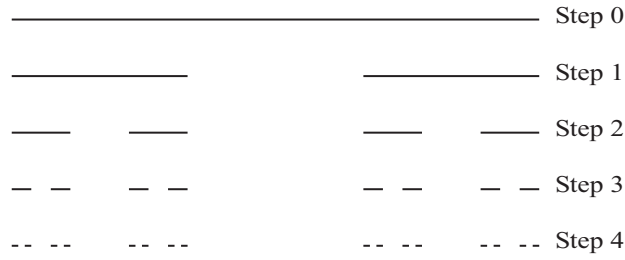


Figure 1: Cantor's set.

e.g., their length, area or volume) only for finite values of  $n$ . The same questions very often remain without any answer when we consider an infinite number of steps because the traditional mathematics (both standard and non-standard versions of Analysis) can speak only about limit fractal objects and the required values tend to zero or infinity.

Let us consider, for example, the famous Cantor's set (see Fig. 1). If a finite number of steps,  $n$ , has been done in construction of Cantor's set, then we are able to describe numerically the set being the result of this operation. It will have  $2^n$  intervals having the length  $\frac{1}{3^n}$  each. Obviously, the set obtained after  $n+1$  iterations will be different and we also are able to measure the lengths of the intervals forming the second set. It will have  $2^{n+1}$  intervals having the length  $\frac{1}{3^{n+1}}$  each. The situation changes drastically in the limit because we are not able to distinguish results of  $n$  and  $n+1$  steps of the construction if  $n$  is infinite.

We also are not able to distinguish at infinity the results of the following two processes that both use Cantor's construction but start from different positions. The first one is the usual Cantor's set and it starts from the interval  $[0, 1]$ , the second starts from the couple of intervals  $[0, \frac{1}{3}]$  and  $[\frac{2}{3}, 1]$ . In spite of the fact that for any given finite number of steps,  $n$ , the results of the constructions will be different for these two processes we have no traditional mathematical tools allowing us to distinguish them at infinity.

Recently a new applied point of view on infinite and infinitesimal numbers (that is not related to the non-standard analysis of Robinson [14]) has been introduced in [16, 18, 19, 20, 21, 22]. The new approach evolves Cantor's ideas about existence of different infinite numbers. It describes infinite and infinitesimal numbers that are in accordance with the principle 'The part is less than the whole' and gives a possibility to work with finite, infinite, and infinitesimal quantities *numerically* by using a new kind of a computer – Infinity Computer – introduced in [15, 17, 18]. A detailed analysis of Cantor's set by using these new tools has been done in [19]. A comprehensive introduction to the new approach and examples of its usage can be found in [17, 20]).

In this paper, we show how the new computational tools can be used to calculate the exact infinitesimal values of area of Sierpinski's carpet and volume of

Menger's sponge. Calculation of numerical characteristics of fractal objects of this kind is particularly important in the context of *E*-Infinity theory (see [1, 2, 4, 3, 7, 8, 9, 10, 11, 23]).

## 2 Physical methodology in Mathematics

In Physics, researchers use tools to describe the object of their study and the used instrument influences results of observations and restricts possibilities of observation of the object. Thus, there exists the philosophical triad – researcher, object of investigation, and tools used to observe the object. A new applied approach to infinity proposed in [16, 20, 21, 22] emphasizes existence of this triad in Mathematics, as well. Mathematical languages (in particular, numeral systems<sup>1</sup>) are among the tools used by mathematicians to observe and to describe mathematical objects. Very often difficulties that we find solving mathematical problems are related not to their nature but to inadequate mathematical languages used to solve them. The new approach is based on the following methodological postulates (see [16, 20, 21]).

**Postulate 1.** *There exist infinite and infinitesimal objects but human beings and machines are able to execute only a finite number of operations.*

**Postulate 2.** *We shall not tell **what are** the mathematical objects we deal with; we just shall construct more powerful tools that will allow us to improve our capacities to observe and to describe properties of mathematical objects.*

**Postulate 3.** *The principle ‘The part is less than the whole’ is applied to all numbers (finite, infinite, and infinitesimal) and to all sets and processes (finite and infinite).*

Due to this declared applied statement, such concepts as bijection, numerable and continuum sets, cardinal and ordinal numbers cannot be used in this paper because they belong to the theories working with different assumptions. As a consequence, the new approach is different also with respect to the non-standard analysis introduced in [14] and built using Cantor's ideas. It is important to emphasize that our point of view on axiomatic systems is also more applied than the traditional one. Due to Postulate 2, mathematical objects are not define by axiomatic systems that just determine formal rules for operating with certain numerals reflecting some properties of the studied mathematical objects.

Due to Postulate 3, infinite and infinitesimal numbers are managed in the same manner as we are used to deal with finite ones. This Postulate in our opinion very well reflects organization of the world around us but in many traditional infinity theories it is true only for finite numbers. Due to Postulate 3, the traditional point of view on infinity accepting such results as  $\infty + 1 = \infty$  are substituted in such a way that  $\infty + 1 > \infty$ .

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<sup>1</sup> We remind that *numeral* is a symbol or group of symbols that represents a *number*. For example, the symbols ‘10’, ‘ten’, and ‘X’ are different numerals, but they all represent the same number.

This methodological program has been realized in [16, 20, 21] where a new powerful numeral system has been developed. This system gives a possibility to execute *numerical* computations not only with finite numbers but also with infinite and infinitesimal ones in accordance with Postulates 1–3. The main idea consists of measuring infinite and infinitesimal quantities by different (infinite, finite, and infinitesimal) units of measure.

A new infinite unit of measure has been introduced for this purpose in [16, 20, 21] in accordance with Postulates 1–3 as the number of elements of the set  $\mathbb{N}$  of natural numbers. It is expressed by a new numeral  $\textcircled{1}$  called *grossone*. It is necessary to emphasize immediately that the infinite number  $\textcircled{1}$  is not either Cantor's  $\aleph_0$  or  $\omega$ . Particularly, it has both cardinal and ordinal properties as usual finite natural numbers. Formally, grossone is introduced as a new number by describing its properties postulated by the *Infinite Unit Axiom* (IUA) (see [16, 20, 21]). This axiom is added to axioms for real numbers similarly to addition of the axiom determining zero to axioms of natural numbers when integer numbers are introduced.

Inasmuch as it has been postulated that grossone is a number, all other axioms for numbers hold for it, too. Particularly, associative and commutative properties of multiplication and addition, distributive property of multiplication over addition, existence of inverse elements with respect to addition and multiplication hold for grossone as for finite numbers. Introduction of grossone gives a possibility to work with finite, infinite, and infinitesimal quantities *numerically* by using a new kind of a computer – the Infinity Computer – introduced in [15, 17, 18].

### 3 Evaluating infinitesimal areas and volumes

Grossone can be successfully used for various purposes related to infinite and infinitesimal objects, particularly, for measuring infinite sets (see [16, 17, 19, 20]). For instance, the number of elements of a set  $B = \mathbb{N} \setminus \{b\}$ ,  $b \in \mathbb{N}$ , is equal to  $\textcircled{1} - 1$  and the number of elements of a set  $A = \mathbb{N} \cup \{a\}$ , where  $a \notin \mathbb{N}$ , is equal to  $\textcircled{1} + 1$ . Note that due to Postulate 3 and definition of  $\textcircled{1}$ , the number  $\textcircled{1} + 1$  is not natural (see [20] for a detailed discussion). Analogously,  $\textcircled{1}^3$  is the number of elements of the set  $W$ , where

$$W = \{(a_1, a_2, a_3) : a_1 \in \mathbb{N}, a_2 \in \mathbb{N}, a_3 \in \mathbb{N}\}. \quad (1)$$

Grossone can be used to indicate positions of elements in infinite sequences, too. Let us investigate this issue in detail. First of all, we remind that the traditional definition of the infinite sequence is: ‘An infinite sequence  $\{a_n\}$ ,  $a_n \in A$  for all  $n \in \mathbb{N}$ , is a function having as the domain the set of natural numbers,  $\mathbb{N}$ , and as the codomain a set  $A$ ’. We have postulated that the set  $\mathbb{N}$  has  $\textcircled{1}$  elements. Thus, due to the sequence definition given above, any sequence having  $\mathbb{N}$  as the domain has  $\textcircled{1}$  elements. Traditionally, the notion of a subsequence is introduced as a sequence from which some of its elements have been cancelled. This definition gives sequences having the number of members (finite or infinite) less than grossone.

Consider, for instance, these two infinite sequences

$$\underbrace{1, 2, 3, 4, \dots, \textcircled{1}-2, \textcircled{1}-1, \textcircled{1}}_{\textcircled{1} \text{ elements}}, \quad \underbrace{3, 4, 5, 6, \dots, \textcircled{1}-2, \textcircled{1}-1}_{\textcircled{1}-3 \text{ elements}}. \quad (2)$$

Thus, the number of elements of any sequence (finite or infinite) is less or equal to  $\textcircled{1}$ . One of the immediate consequences of the understanding of this result is that any sequential process can have at maximum  $\textcircled{1}$  steps. It is very important to notice a deep relation of this observation to the Axiom of Choice. Since *any* sequential process can have at maximum  $\textcircled{1}$  elements, this is true for the process of choice, as well. Therefore, it is not possible to choose in a sequence more than  $\textcircled{1}$  elements from a set. For example, if we consider the following set

$$D = \{1, 2, 3, 4, \dots, \textcircled{1}-2, \textcircled{1}-1, \textcircled{1}, \textcircled{1}+1, \textcircled{1}+2, \dots, \textcircled{1}^3-2, \textcircled{1}^3-1, \textcircled{1}^3\}$$

and a process of the sequential choice of elements from this set in the order of increase of their values, it is possible to arrive at maximum to  $\textcircled{1}$  starting from 1

$$\underbrace{1, 2, 3, 4, \dots, \textcircled{1}-2, \textcircled{1}-1, \textcircled{1}}_{\textcircled{1} \text{ steps}}, \textcircled{1}+1, \textcircled{1}+2, \dots, \textcircled{1}^3-2, \textcircled{1}^3-1, \textcircled{1}^3 \quad (3)$$

executing so  $\textcircled{1}$  steps. Starting from 3 it is possible to arrive at maximum to  $\textcircled{1}+2$

$$1, 2, 3, 4, \dots, \underbrace{\textcircled{1}-2, \textcircled{1}-1, \textcircled{1}, \textcircled{1}+1, \textcircled{1}+2}_{\textcircled{1} \text{ steps}}, \textcircled{1}+3, \dots, \textcircled{1}^3-2, \textcircled{1}^3-1, \textcircled{1}^3. \quad (4)$$

Of course, due to Postulate 1, we are able to observe only a finite number among  $\textcircled{1}$  members of these processes. In addition, it depends on the chosen numeral system,  $\mathcal{S}$ , which numbers we can observe because  $\mathcal{S}$  should have numerals able to represent the required numbers.

Another important observation consists of the fact that numeral systems including  $\textcircled{1}$  allow us to observe the starting and the *ending* elements of infinite processes, if the respective elements are expressible in these numeral systems. This fact is very important in connection with fractals because it allows us to distinguish different fractal objects after an infinite number of steps of their construction. This observation concludes preliminary results and allows us to start to study Sierpinski's carpet (see Fig. 2).

Since construction of Sierpinski's carpet is a process, it cannot contain more than  $\textcircled{1}$  steps (see discussion related to (2)-(4)). Thus, if at iteration  $n = 1$  the process starts from the first from the left grey square having the length of a side equal to 1, at iteration  $n \geq 1$  Sierpinski's carpet consists of  $N_n = 8^{n-1}$  grey boxes. The length of a side of a hole,  $L_n$ , is equal to  $3^{-(n-1)}$  and the area of all grey boxes is equal to

$$A_n = L_n^2 \cdot N_n = \left(\frac{8}{9}\right)^{n-1}. \quad (5)$$

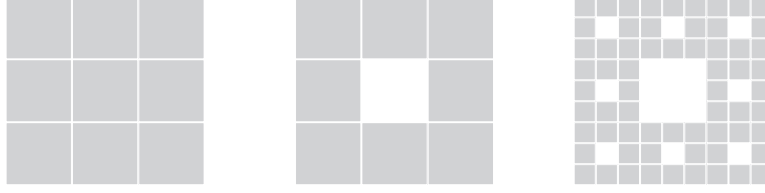


Figure 2: Sierpinski's carpet.

Therefore, if ① steps have been executed, the area  $A_{\textcircled{1}} = \left(\frac{8}{9}\right)^{\textcircled{1}-1}$  and if ①-9 steps have been executed, the area  $A_{\textcircled{1}-9} = \left(\frac{8}{9}\right)^{\textcircled{1}-10}$ . It is worthwhile to notice that (again due to the limitation illustrated by the example (2)-(4)) it is not possible to count one by one all the boxes at Sierpinski's carpet if their number is superior to ①. For instance, at iteration ① it has  $8^{\textcircled{1}-1}$  boxes and they cannot be counted sequentially because  $8^{\textcircled{1}-1} > \textcircled{1}$  and any process (including that of the sequential counting) cannot have more than ① steps.

Thus, we are able now to distinguish different Sierpinski's carpets at different points at infinity and to calculate the respective areas that are expressed in infinitesimals. Moreover, we can do it also when we change the starting element of the construction. For instance, if at iteration  $n = 1$  the process starts from the second from the left object consisting of 8 grey squares having the length of a side equal to  $\frac{1}{3}$ , at iteration  $n \geq 1$  Sierpinski's carpet consists of  $N_{2,n} = 8^n$  grey boxes, where the subscript indicates the numbers of the starting and the ending points of the process. The length of a side of a hole  $L_{2,n} = 3^{-n}$  and the area of all grey boxes is equal to

$$A_{2,n} = L_{2,n}^2 \cdot N_{2,n} = \left(\frac{8}{9}\right)^n. \quad (6)$$

Then, if ① steps have been executed, the area  $A_{2,\textcircled{1}} = \left(\frac{8}{9}\right)^{\textcircled{1}}$  and if ①-9 steps have been executed, the area  $A_{2,\textcircled{1}-9} = \left(\frac{8}{9}\right)^{\textcircled{1}-9}$ . Obviously, formulae (5), (6) can be easily generalized to the case where  $N_{k,n} = 8^{n+k-2}$ ,  $L_{k,n} = 3^{-(n+k-2)}$  and

$$A_{k,n} = L_{k,n}^2 \cdot N_{k,n} = \left(\frac{8}{9}\right)^{n+k-2}, \quad 1 \leq k \leq n \leq \textcircled{1} + k - 1, \quad (7)$$

and (7) can be used both for finite and infinite values of  $k$  and  $n$ .

We conclude this paper by evaluating the volume of Menger's sponge by a complete analogy to (5)–(7). Let  $n$  be the number of iterations. Then, starting from the left cube at iteration  $n = 1$  the number of grey cubes is  $N_{1,n} = 20^{n-1}$ ,  $n \geq 1$ . The length of a side of a hole  $L_{1,n}$  is equal to  $3^{-(n-1)}$  and the volume of all grey cubes at the  $n$ -th iteration is equal to

$$V_{1,n} = L_{1,n}^3 \cdot N_{1,n} = \left(\frac{20}{27}\right)^{n-1}. \quad (8)$$

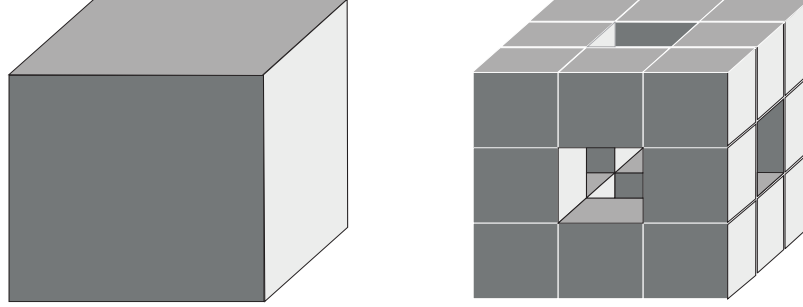


Figure 3: Menger's sponge.

For  $n = \textcircled{1}$  we have  $V_{1,\textcircled{1}} = \left(\frac{20}{27}\right)^{\textcircled{1}-1}$  and for  $n = \textcircled{1} - 1$  it follows  $V_{1,\textcircled{1}-1} = \left(\frac{20}{27}\right)^{\textcircled{1}-2}$ . Finally, the general formula of the volume of Menger's sponge is

$$V_{k,n} = L_{k,n}^3 \cdot N_{k,n} = \left(\frac{20}{27}\right)^{n+k-2}, \quad 1 \leq k \leq n \leq \textcircled{1} + k - 1. \quad (9)$$

## 4 Conclusion

Very often traditional approaches studying dynamics of self-similarity processes are not able to give their quantitative characteristics at infinity and, as a consequence, use limits to overcome this difficulty. For example, it is well known that the limit area of Sierpinski's carpet and volume of Menger's sponge are equal to zero.

In this paper, it has been shown that infinite and infinitesimal numbers introduced in [16, 18, 19, 20, 21, 22] allow us to obtain exact numerical results instead of limits and to calculate exact infinitesimal values of areas and volumes of various fractal objects at different points at infinity. In fact, the possibility to express explicitly various infinite numbers allows us to indicate the final elements not only for finite but for infinite processes, as well. As a result, we can calculate the required areas and volumes at the chosen moment even if this moment is infinitely faraway on the time axis from the starting point. It is interesting that traditional results that can be obtained without the usage of infinite and infinitesimal numbers can be produced just as finite approximations of the new ones.

It has been shown the importance of the starting conditions in fractal processes. Suppose that there are two identical fractal processes starting from different starting structures in such a way that the starting structure of the second process is a result of  $k$  steps of the first one. Then, after the same number of steps,  $n$ , the two processes lead to different results for both finite and infinite values of  $n$ . Thus, this fact being obvious for finite values of  $n$  holds for infinite values of  $n$ , too. Finally, the importance of the possibility to have this kind of quantitative characteristics for *E*-Infinity theory (see [1, 2, 4, 3, 7, 8, 9, 10, 11, 23]) has been emphasized in the paper.

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